THE SKOROKHOD PROBLEM STUDY VIA DESYNCHRONIZED SYSTEMS TECHNIQUES

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Abstract

It is shown that the techniques developed as a tool for the stability analysis of the so called desynchronized discrete event systems may be successfully applied to the investigation of the Skorokhod problem on an orthant in $\mathbb{R}^N$. As a result, new conditions for the uniqueness of the Skorokhod problem solution and for the Lipschitz continuity of the corresponding oblique projection mapping are obtained. The robustness of non-uniqueness of the Skorokhod problem solution as well as the robustness of the Lipschitz continuity for the corresponding oblique projection mapping are also established.

Keywords: Skorokhod problem, Lipschitz continuity, stability, desynchronized systems, controllability, robustness.

Introduction

Recently much attention was paid to the development of methods for the analysis of dynamics of multicomponent systems with asynchronously interacting subsystems (see [1, 3, 7] for further references). As examples we can mention the systems with faults in data transmission channels, multiprocessor computing and telecommunication systems, flexible manufacturing systems and so on. It turned out that under weak and natural assumptions systems of this kind possess such a strong property as robustness. In applications the robustness is often treated as reliability of a system with respect to perturbation of various nature, e.g., drift of parameters, malfunctions or noises in data transmission channels, etc.

This paper was partially stimulated by fruitful discussions with A.Mandelbaum (Israel) concerning the unique solvability of the dynamic complementarity problem that is also known as the Skorokhod problem or the problem on processes with oblique reflections. These problems arise in the system theory (properties of transducers with hysteresis nonlinearities), stochastic flow networks, Brownian motion in bounded domains, mechanics of elastic-plastic constructions, market economy, theory of partial differential equations, etc. (see, e.g., [4, 5, 6]).

The close relation of the stability problem for desynchronized systems with the Skorokhod problem was found. This relation presents an opportunity to use the techniques of desynchronized systems study for investigation of the Skorokhod problem and vice versa. The aim of this paper is to describe the relations mentioned above and to derive some corollaries of these relations.

1 Desynchronized systems

Let us introduce basic notions of the desynchronized systems theory. Consider a linear system S consisting of N subsystems S_1, S_2, ...,S_N that interact at some discrete instants $\{T^n\}$, $-\infty < n < \infty$. The moments of interaction may be chosen according to some deterministic or stochastic law but generally they are not known in advance. Let the state of each subsystem S_i be determined within the interval $[T^n, T^{n+1})$ by a numerical value $x_i(n)$, $-\infty < n < \infty$.

Let at each moment $T^n \in \{T^k : \quad -\infty < k < \infty\}$ only one of subsystems S_i (say, with an index $i = i(n) \in \{1, 2, \ldots, N\}$) may change its state and the law of the state updating be linear:

$$x_i(n+1) = \sum_{i=1}^{N} a_{ij} x_j(n), \qquad i = i(n).$$

Consider the matrix $A = (a_{ij})$ and introduce for each $i = 1, 2, \ldots, N$ an auxiliary matrix A_i (*i-mixture* of the matrix A) that is obtained from A by replacing its rows with indexes $i \neq j$ with the corresponding rows of the identity matrix I. Then the dynamics equation for the system S can be written in the following compact form:

$$x(n+1) = A_{i(n)} x(n), \qquad -\infty < n < \infty. \tag{1}$$

The system described above is referred to as *the linear desynchronized or asynchronous* *system.* Desynchronized systems of more general nature and under more general assumptions were considered in [1, 8].

1.1 Stability of desynchronized systems

Traditionally the stability is recognized as the most important intrinsic property of a system. As was already mentioned, the index sequence $\{i(n)\}$ in (1) may be unknown. This leads us naturally to the following notion.

Definition 1.1 *A desynchronized system S is called absolutely stable (AS) if for any index sequence $\{i(n)\}$ each solution $x(n)$ of corresponding equation (1) is bounded for $n \geq 0$.*

An index sequence $\{i(n)\}$ with infinitely many occurencies of each number $1, 2, \ldots, N$ is called *regular*.

Definition 1.2 *A desynchronized system S is called absolutely asymptotically stable (AAS) if it is AS and for any regular index sequence $\{i(n)\}$ each solution $x(n)$ of (1) tends to zero as $n \to \infty$.*

A system S is called Perron stable (PS) if for any index sequence $\{i(n)\}$ and for any sequence of vectors $b(n)$ bounded for $n \geq 0$, the solution $x(n)$ of the inhomogeneous equation

$$x(n+1) = A_{i(n)}x(n) + (I - Z_{i(n)})b(n),$$

satisfying initial condition $x(0) = 0$ (here Z_i denotes a corresponding mixture of zero matrix Z) is also bounded for $n \geq 0$.

Some classes of *AS* and *AAS* desynchronized systems were studied in [1, 3, 7, 8]. The same publications contain variety of necessary, sufficient, necessary and sufficient conditions of *AS* and *AAS*.

Theorem 1.3 *The AAS and PS properties for linear desynchronized systems are equivalent.*

Thus, according to Theorem 1.3, both cases in Definition 1.2 deal with the same kind of stability. The differences in formulations present good opportunities in investigation of the stability of desynchronized systems.

A variety of *AS* conditions for desynchronized systems can be obtained with the help of equivalent norm techniques (see, e.g., [1, 3, 7, 8, 9]) that is based on the following statement. *A desynchronized system possesses the AS property, iff there exists a norm $\|\cdot\|_*$ such that $\|A_i\|_* \leq 1$, $i = 1, 2, \ldots, N$.* Analogous criteria for *AAS* and *PS* in terms of equivalent norms with some additional properties are also known (see, e.g., [8]).

1.2 Quasi-controllability and robustness of desynchronized systems

One of a most important properties of any system modeling real phenomena is the robustness of its qualitative behavior with respect to small deviations of its parameters. As is easily seen, generally the AS property is not robust. At the same the following result is valid.

Theorem 1.4 *The AAS and PS properties for a linear desynchronized system are robust with respect to small perturbation of entries of the matrix A.*

As it turned out, robustness of instability of desynchronized systems is essentially affected by such a property of a system as its *quasi-controllability* [10].

Definition 1.5 *A desynchronized system S is called quasi-controllable if there is no nontrivial proper subspace* $L \subseteq \mathbb{R}^N$ *that is invariant for matrixes* $A_1, A_2, \ldots, A_N$.

It is very easy to verify quasi-controllability of a certain desynchronized system S as follows. The system S is quasi-controllable, iff 1 is not an eigenvalue of the matrix A and the matrix A is irreducible, i.e., it cannot be represented in a block triangular form by swapping some of its rows and corresponding columns.

We shall say that a system S possess the $\overline{AS}$ property if it does not possess the AS property.

Theorem 1.6 *Property* $\overline{AS}$ *for a linear desynchronized quasi-controllable system is robust with respect to small perturbation of entries of the matrix A.*

We derive next two corollaries of this theorem. One of them is important for the understanding of a structure of the set of AS desynchronized systems and another is essential in establishing relations between desynchronized systems and the Skorokhod problem.

Corollary 1.7 *The set of AS linear desynchronized quasi-controllable systems is closed if the proximity of systems is treated as the proximity of corresponding matrixes A.*

Corollary 1.8 *If a linear desynchronized system S possesses the property* $\overline{AS}$*, then there exists an index sequence* $\{i(n)\}$ *such that there is an exponentially increasing solution of (1):*

$$\|x(n)\| \geq \lambda^n \|x(0)\|, \qquad (n \geq 0),$$

where $\lambda > 1$, $x(0) \neq 0$.

2 Skorokhod problem

Let the state of a controlled system S at each moment t be uniquely determined by an input (control) vector $u(t) \in \mathbb{R}^N$ and an output vector $x(t) \in \mathbb{R}^N$. The output of the system is assumed to be restricted by condition $x(t) \in G \subset \mathbb{R}^N$, $t \geq 0$, where G is a closed convex set (we consider further only the case $G = \mathbb{R}^N_+$, where $\mathbb{R}^N_+$ is a positive orthant in $\mathbb{R}^N$).

The output $x(t)$ inside G behaves in the same way as the input $u(t)$, i.e., $x(t) = x(t_0) + u(t) - u(t_0)$ while $x(t_0) + u(t) - u(t_0) \in G$. The behavior of $x(t)$ on the boundary of G is determined by the special "reflection law"; for example, in [14, 12, 13] the case of the "normal reflection" is considered, i.e.,

$$x(t) = x(t_0) + u(t) - u(t_0) + \int_{t_0}^{t} y(\tau)d\tau$$

where the unknown vector $y(\tau)$, called a *regulation vector*, belongs to the inward normal cone of G at the point $x(\tau)$. More general problems where considered in [5], the paper [6] deals with the case of an "oblique" reflection law determined on each face of the orthant $\mathbb{R}^N_+$ by the *generalized normal vector* q^i. These problems are called *Skorokhod problem* (SP) or *Dynamic complementarity problem* (when $G = \mathbb{R}^N_+$).

Introduce a formal definition of the SP. Denote by $C([0,T];\mathbb{R}^N)$ the space of continuous functions from $[0,T]$ to $\mathbb{R}^N$ and by $D([0,T];\mathbb{R}^N)$ the space of right-continuous functions with left limits on $[0,T]$. Let $|v|(T)$ denote the total variation of function v on $[0,T]$. For a function of bounded variation $x(t) : [0,T] \to \mathbb{R}^N$ we shall denote by $d|x|(t)$ the corresponding Lebesgue-Stieltjes measure on $[0,T]$. Function $x(t)$ is absolutely continuous with respect to this measure and it has a $d|x|(t)$-measurable density function uniquely determined $d|x|(t)$-almost everywhere by the equality $\gamma(t) = dx(t)d|x|(t)$.

Let G be as above and the reflection cone $d(x)$ be defined for each $x \in G$ such that $d(x) = \{0\}$ whenever $x \in \text{int}G$.

Definition 2.1 (Skorokhod Problem) *Let $u \in D([0,T];\mathbb{R}^N)$ with $u(0) \in G$ be given. The triple $\{x,u,y\}$ is called solution of the SP (with respect to G and $d(x)$) if*

- *$x = u + y$, $x(0) = u(0)$, $x(t) \in G$ for $t \in [0,T]$,*
- *there exists measurable $\gamma : [0,T] \to \mathbb{R}^N$ such that $\gamma(s) \in d(x(s))$ for all $s \in [0,T]$ and*

$$y(t) = \int_{(0,t]} \gamma(s)d|y|(s), \qquad |y(T)| < \infty.$$

The corresponding mapping $u \to x$ is called a Skorokhod mapping.

2.1 Uniqueness of the Skorokhod problem solution

One of a most interesting and most difficult questions related to the Skorokhod problem is the following. What are conditions that should be imposed on the generalized normal vectors to ensure the uniqueness of the output $x(t)$ for any continuous (or more general) input $u(t)$?

Definition 2.2 *A SP is called uniquely solvable if it has exactly one solution for any input $u \in D([0,T];\mathbb{R}^N)$, $u(0) \in G$.*

For the case $G = \mathbb{R}_+^N$ considered further we shall assume $d(x) = \text{cone}\{q^i, i \in I(x)\}$, where $I(x) = \{i : x_i = 0\}$. We shall suppose that $q_i^i > 0$, $i = 1, 2, \ldots, N$. Let us associate with the SP considered above the $N\times N$-matrix $Q = \{q_{ij} = q_i^j\}$, $i, j = 1, 2, \ldots, N$. It is convenient to denote the above SP by $\mathcal{S}(Q)$. As is known (see, e.g., [5]), the problem $\mathcal{S}(Q)$ has a solution for every input $u(t) \in D([0,T];\mathbb{R}^N)$, $u(0) \in G$, if and only if Q is a completely-S matrix (remind that a matrix Q is called *a S-matrix* if there exists a vector $\eta \geq 0$ such that $Q\eta > 0$; Q is *completely-S* if all its principal submatrixes are S-matrixes). In [6] the first sufficient conditions for the uniqueness of the $\mathcal{S}(Q)$ solution were obtained: *the matrix $I-Q$ should be nonnegative with a spectral radius less then 1.* The next statement follows from the results in [2, 11]: *if $\mathcal{S}(Q)$ is uniquely solvable, then Q is a P-matrix, i.e., all its principal minors are strictly positive.* We shall assume further that Q is a P-matrix.

The review of some recent results concerning the solvability of the SP and the Lipschitz continuity of the Skorokhod mapping may be found in the paper [5]. Our aim is to formulate necessary and sufficient conditions for the uniqueness of the Skorokhod problem solution and for the Lipschitz continuity of the Skorokhod mapping in terms of the stability of certain desynchronized system. To this end, let us associate with the matrix Q an auxiliary matrix

$$A = I - \text{diag}(q_{11}, q_{22}, \ldots, q_{NN})^{-1}Q. \tag{2}$$

Theorem 2.3 *The SP with an irreducible P-matrix Q is uniquely solvable iff the desynchronized system with the matrix (2) is absolutely stable.*

2.2 Lipschitz continuity of Skorokhod mapping

The Lipschitz continuity of the Skorokhod mapping is important in many situations (see, e.g., [5] as an example of utilization of this property for obtaining a large deviation type results and others). The Lipschitz continuity of SP on an arbitrary polyhedral domain G with the normal law of reflection was first proved in [14] and recently rediscovered in [5].

Definition 2.4 *The Skorokhod mapping is called Lipschitz continuous if the SP is uniquely solvable and there exists $C > 0$ such that for any pair of inputs u_1, u_2 satisfying $\|u_1(t) - u_2(t)\| \leq 1$ $(0 \leq t \leq T)$ the following inequality is fulfilled: $\|x_1(t) - x_2(t)\| \leq C$ $(0 \leq t \leq T)$ (here x_i is the solution of SP for the input u_i).*

Theorem 2.5 *The Skorokhod mapping for the SP with an irreducible P-matrix Q is Lipschitz continuous iff the desynchronized system with the matrix (2) is absolutely asymptotically stable.*

3 Corollaries and generalizations

Theorem 3.1 *The SP with irreducible P-matrix Q is uniquely solvable if and only if the SP with the transposed matrix Q' is uniquely solvable.*

Theorem 3.2 *Both the properties of non-unique solvability and Lipschitz continuity of the SP with irreducible P-matrix Q are robust with respect to small perturbation of Q.*

Theorem 3.3 *Let $U = I - A$, where A is irreducible matrix (2). If all the eigenvalues of the symmetric matrix $V = \frac{1}{2}(U + U^T)$ belong to the interval $[\rho, 2 - \rho]$, $0 < \rho \leq 1$, and the spectral radius r of the skew-symmetric matrix $W = \frac{1}{2}(U - U^T)$ satisfies the equation*

$$r < \rho\sqrt{\frac{1-\rho}{1+\rho}}\left\{\frac{1}{\sqrt{1-(1-\rho^2)^N}} - 1\right\},$$

then the problem $\mathcal{S}(Q)$ is uniquely solvable.

Theorems of this section follow from the results of desynchronization theory. All the results of this paper may be generalized to the case of a polyhedral domain.

References

[1] E.A. Asarin, V.S. Kozyakin, M.A. Krasnosel'skii, and N.A. Kuznetsov. Stability analysis of desynchronized systems. In *Preprints of the 11th IFAC World Congress*, Tallinn, Estonia, USSR, Vol. 2:56–60, 1990.

[2] A. Berman and R.J. Plemmons. *Non-Negative Matrices in the Mathematical Sciences.* Academic Press, N.-Y., 1979.

[3] D.P. Bertsekas and J.N. Tsitsiklis *Parallel and Distributed Computation. Numerical Methods.* Englewood Cliffs, NJ, 1988.

[4] H. Chen and A. Mandelbaum. Stochastic discrete flow networks: diffusion approximations and bottlenecks. *The Ann. Probab.*, 19(4):1463–1519, 1991.

[5] P. Dupuis and H. Ishii. On lipschitz continuity of the solution mapping to the skorokhod problem with applications. *Stoch. and Stoch. Rep.*, 35:31–62, 1991.

[6] J.M. Harrison and M.I. Reiman. Reflected brownian motion on an orthant. *Annals of Probability*, 9:302–309, 1981.

[7] A.F. Kleptsyn, V.S. Kozyakin, M.A. Krasnosel'skii, and N.A. Kuznetsov. Stability of desynchronized systems. *Dokl. Akad. Nauk USSR*, 274(5):1053–1056, 1984. (Russian).

[8] V.S. Kozyakin. Equivalent norms technique for stability analysis of linear discrete event systems. In *Proceedings of the 1991 IFAC Workshop on Discrete Event System Theory and Applications in Manufacturing and Social Phenomena, June 25-27, 1991, Shenyang, China*, pages 290–295, Internat. Acad. Publishers, a Pergamon-CNPIEC Joint Venture, 1991.

[9] V.S. Kozyakin, On perturbations of linear desynchronized systems. *Doklady AN SSSR*, 316(1):54-57, 1991. (Russian).

[10] V.S. Kozyakin and A.V.Pokrovskii. The role of the controllability-type properties for the investigation of the stability of the desynchronized dynamical systems. *Doklady AN SSSR*, 324(1):60–64, 1992. (Russian).

[11] J. Samuelson, R.M. Thrall, and O. Wesler. A partition theorem for Euclidean N-space. *Proc. Am. Math. Soc.*, 9:805–807, 1958.

[12] H. Tanaka. Stochastic differential equations with reflecting boundary condition in convex region. *Hiroshima Math. J.*, 9:163–177, 1979.

[13] A.A. Vladimirov. Averaging properties of multidimensional hysteresis operators. *Dokl. Akad. Nauk USSR*, 309:1289–1292, 1989. (Russian).

[14] A.A. Vladimirov and A.F. Kleptsyn. On some hysteresis elements. *Avtom. i Telemekh.*, (7):165–169, 1982. (Russian).